# Quantum Kerr Learning 

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(Dated: May 25, 2022)


#### Abstract

Quantum machine learning is a rapidly evolving area that could facilitate important applications for quantum computing and significantly impact data science. In our work, we argue that a single Kerr mode might provide some extra quantum enhancements when using quantum kernel methods based on various reasons from complexity theory and physics. Furthermore, we establish an experimental protocol, which we call quantum Kerr learning based on circuit QED. A detailed study using the kernel method, neural tangent kernel theory, first-order perturbation theory of the Kerr non-linearity, and non-perturbative numerical simulations, shows quantum enhancements could happen in terms of the convergence time and the generalization error, while explicit protocols are also constructed for higher-dimensional input data.


Introduction.-Quantum machine learning, i.e., combining machine learning with the computational power of quantum devices, is an exciting, emerging direction for modern information technology [1-4]. In the present Noisy Intermediate-Scale Quantum (NISQ) era [5], there has been significant progress on quantum machine learning based on variational quantum circuits 6-15. However, it is still not completely clear, both in theory and practice, if and how a true quantum advantage relative to completely classical-computer based approaches will be achieved 15, 16.

The kernel methods developed in machine learning over many years [17] could have natural connections to quantum mechanics. In the spirit of a kernel method, nonlinear machine learning problems could be transformed towards linear models in a sufficiently high dimensional Hilbert space. In the classical machine learning context, the corresponding Hilbert space of a given kernel is abstract, while in the quantum computing setup, the Hilbert space could be physical, such as the state vector space of a quantum device [18]. Complex quantum systems could produce complicated enough kernels that are hard to simulate using classical computers, providing a potential regime of quantum advantage in quantum machine learning 3].

Thus, how could we produce complicated kernels in a quantum system? In the quantum kernel method, the kernel is evaluated from quantum measurement. For instance, if we consider a Hamiltonian with the real time evolution, $H(t)$, one could construct the kernel as

$$
\begin{align*}
K= & \left\lvert\,\langle\psi| \overline{\mathbb{T}} \exp \left[\frac{i}{\hbar} \int_{0}^{T^{\prime}} H\left(t^{\prime}\right) d t^{\prime}\right]\right. \\
& \times\left.\mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} H(t) d t\right]|\psi\rangle\right|^{2} . \tag{1}
\end{align*}
$$

Here, we are measuring the inner product of two quantum states, starting from $|\psi\rangle$, and evolving with the Hamiltonian $H(t)$ during time $T$ and $T^{\prime}$ respectively, where $\mathbb{T}$ denotes time ordering in quantum mechanics. The more complicated the Hamiltonian is, the harder it is for a classical computer to simulate. Here, the kernel $K$ is a matrix where the matrix element is specified by the input data $\mathbf{x}$ and $\mathbf{x}^{\prime}$. The vectors $\mathbf{x}$ or $\mathbf{x}^{\prime}$ could be made from parameters of Hamiltonians or the evolution time.

Model setup. -In this paper, we consider a single-mode quantum system with Kerr non-linearity that could potentially provide computational benefits compared to its classical counterparts. Such systems can be easily realized in circuit QED experiments where the Kerr nonlinearity can be obtained from Josephson junctions or the kinetic inductance of a superconducting resonator [19, 20. We propose and describe the experimental realization in the Supplemental Material (SM). The effective Hamiltonian of the system is

$$
\begin{align*}
& H=H_{0}+H_{\mathrm{t}}+H_{\mathrm{I}}=H_{\mathrm{f}}+H_{\mathrm{I}} \\
& \frac{H_{0}}{\hbar}=\omega_{\mathrm{m}}\left(b^{\dagger} b+\frac{1}{2}\right), \frac{H_{\mathrm{t}}}{\hbar}=\Omega\left(b e^{-i \omega_{\mathrm{L}} t}+b^{\dagger} e^{i \omega_{\mathrm{L}} t}\right) \\
& \frac{H_{\mathrm{I}}}{\hbar}=-K_{\mathrm{err}} b^{\dagger} b^{\dagger} b b, H_{\mathrm{f}}=H_{0}+H_{\mathrm{t}} \tag{2}
\end{align*}
$$

We define the real vector $\mathbf{x}=(\mathbf{x})_{i=1}^{3}=\left(\Omega, \omega_{\mathrm{L}}, T\right)$ as the encoded data for simplicity (namely, our supervised learning task has three-dimensional inputs), and we fix $\left(\omega_{\mathrm{m}}, K_{\text {err }}\right)$ as constants. When the Kerr nonlinearity is turned off, the system Hamiltonian reduces to a quadratic form in the rotating frame. If we turn on the Kerr non-linearity and when $K_{\text {err }}$ is much smaller than $\Omega$, the kernel could be predicted by perturbation theory (see SM for detailed calculations). On the other hand, when $K_{\text {err }}$ is much larger than $\Omega$, the non-linear
resonator can be approximated as a qubit with negligible excitations at energy levels higher than the first excited state (see SM). However, for a general $K_{\text {err }}$ in the regime of $K_{\text {err }} / \Omega \sim \mathcal{O}(1)$, we do not have analytic predictions and the system must be simulated by quantum or classical computers.

We argue that quantum enhancements could be achieved from our Kerr non-linear quantum system (here, the word "quantum enhancements" indicate benefits in computation provided by the Kerr non-linearity, which are not the same as "quantum advantage" claimed by the rigorous complexity theory statements). We give the following four reasons.

First, a general real-time evolution of a quantum system could be formulated in the complexity class BQP, which is the class of problems solved by quantum computers with bounded error and polynomial time. More precisely, the problem is BQP-complete. As long as BQP is not equal to $\mathbf{P}$, the problem will be hard to simulate in general for classical computers [21-23]. In the bosonic system, the quadratic Hamiltonian is simulatable by classical computers in polynomial time, based on the celebrated Gottesman-Knill theorem [24] in the bosonic case with the Gaussian initial states [25]. Thus, since our Hamiltonian includes Kerr non-linearity, by the inverse statement of the bosonic Gottesman-Knill theorem, we suggest that our Hamiltonian Eq. (2) might be challenging to simulate for classical computers for large scale simulations, and for our analog quantum system, one could evaluate the kernel efficiently within the polynomial time.

Second, the physical systems will naturally set the ultraviolet cutoff by the energy scale of the devices, and we do not need to set the cutoff scale $\Lambda$ just like what one did in classical simulations. Classically we have to truncate the Hamiltonian of the bosonic system to ensure that the cutoff is sufficient where interesting non-linear physics will happen and will accurately predict the real dynamics, and there will be limited principles to determine the proper range of $\Lambda$ because of our computational power. However, in our circuit QED system, the system naturally provides the cutoff by the energy scale of photons and electrons, so we could get quantum enhancement there, especially when the required $\Lambda$ is large.

Third, although in the bulk of our paper, we keep $\Omega$ as a constant, we could also make it time-dependent. For the time-dependent $\Omega(t)$, it is shown that the Hamiltonian in Eq. (2) could simulate universal quantum computing in the polynomial time, and the whole class BQP [26], even for small Kerr couplings. Thus, we expect that a combination of $\Omega$ and $K_{\text {err }}$ will provide quantum enhancements.

Fourth, although the bulk of our paper is about single modes, we briefly give some statements about multimodes. A multimode Hamiltonian in our circuit QED setup will include several free bosonic modes that are
interacting with another Kerr mode. Through the coupling, those free modes will have more complicated dynamics, and we expect that it might present stronger quantum enhancements. An analog of the multiple mode system, will be a zero-dimensional $\lambda \phi^{4}$ quantum field theory on the lattice with the system size $N$ (note that the Kerr term is also quartic), where we have the theoretical predictions when the bare coupling $\lambda$ is either zero or infinity. The non-perturbative corrections will happen around the critical point, and the strong coupling regime, where we have a $\mathbb{Z}_{2}$ symmetry which is spontaneously broken, and we expect a quantum speedup to simulate the non-perturbative dynamics [27] 29 .

In this paper, further solid theoretical and numerical evidence are given to show that larger Kerr non-linearity will provide more significant enhancement of quantum kernels. We qualify the performance of kernels based on the theory of the kernel method [17] and the neural tangent kernel theory for linear models [15, 30-37]. Better kernels will have flatter eigenspectra with more non-trivial kernel eigenvalues, which will lead to faster convergence speed [37] (see discussions in SM) and less generalization error for good enough alignments [38,42]. These features are observable through the numerical results in the kernel trick and the gradient descent dynamics. We find that non-trivial $K_{\text {err }}$ might generically lead to better performance in all those criteria, at least in our data distribution within the allowed experimental range. A by-product of our work is that we compute the leading order perturbation of our kernel function $K$ (see SM). Since perturbation theory gives closed-form formulas, we could evaluate the kernel efficiently through analytic formulas, instead of numerical simulations of our bosonic quantum systems up to a given truncation. Our perturbation theory formula is well-tested through numerics and is valid in the perturbative range of $K_{\text {err }}$.

Kernel Statistics.-In our analysis, we randomly generate the training input data $\mathbf{x}=\left(\Omega, \omega_{\mathrm{L}}, T\right)$ through uniform distributions within experimentally feasible data range. We set the data range from $(0,0,0)$ to $\left(\Omega^{\text {range }}, \omega_{\mathrm{L}}^{\text {range }}, T^{\text {range }}\right) \sim(300 \mathrm{MHz} \times 2 \pi, 10 \mathrm{GHz} \times$ $2 \pi, 0.05 \mu \mathrm{~s})$. Moreover, we fix $\omega_{\mathrm{m}}$ at around $10 \mathrm{GHz} \times 2 \pi$ for a typical superconducting microwave resonator.

The property of the kernel function $K$, or more precisely the neural tangent kernel (NTK) $K_{\mathrm{H}}=K^{2}$ of the linear model (see SM), plays an important role in the performance of the machine learning algorithm. In Fig. 1. we study the change of the kernel eigenspectra when turning on the Kerr coupling. We measure the complexity of the kernel by looking at the kernel effective dimension (number of eigenvalues that are not small, where we set the criterion to be $>10^{-7}$ ) and the maximal kernel eigenvalues. As shown in Fig. 1, we find that generically when we turn on the Kerr coupling towards the nonperturbative regime, the kernel spectra have the tendency to be flatter, inferred from the increasing kernel effective
dimension. This suggests that better performance can be achieved in generic numerical optimization experiments for larger Kerr non-linearity. In fact, a flatter spectrum means that the distribution has longer tails towards the higher values of the NTK eigenvalue parameters, which might indicate faster convergence and less generalization error for sufficient alignments.

This tendency could even be observed in the perturbative regime, where the kernel properties are predictable through the perturbation theory. We verify the validity of our perturbation theory prediction, comparing to the numerical simulation for truncated Hilbert space dimensions, in Fig. 2.


FIG. 1. Kernel statistics when turning on Kerr non-linearity. We generate $\mathcal{O}(100)$ data vectors when evaluating the kernel statistics using the perturbation theory (red), and $\mathcal{O}(10)$ data vectors when evaluating the kernel statistics using the numerical exact simulation with 100 truncated bosonic energy levels (blue). As evidence of where nontrivial Kerr coupling complexifies the kernel, we count the number of eigenvalues of $K_{\mathrm{H}}$ that are larger than $10^{-7}$ (the kernel effective dimension) when turning on the Kerr coupling (up), and evaluate the maximal eigenvalues of $K_{\mathrm{H}}$ depending on $K_{\text {err }}$. Note that when $K_{\text {err }} \geq \mathcal{O}(10) \mathrm{MHz} \times 2 \pi$, we start entering the regime of non-perturbative dynamics where perturbative theory predictions may not be trustable.


FIG. 2. The statistics of the relative error of matrix elements $\left|\left(K_{\mathrm{t}}-K_{\mathrm{e}}\right) / K_{\mathrm{e}}\right|$ for the perturbation theory prediction of the kernel $K_{\mathrm{t}}$ and the numerical simulation $K_{\mathrm{e}}$ among $\mathcal{O}(100)$ randomly generated data inputs $\mathbf{x}$ (the red points). We set $K_{\text {err }}=\mathcal{O}(0.01) \mathrm{MHz} \times 2 \pi$.

Optimization Simulations.-The property of kernel statistics could be directly examined through the actual gradient descent dynamics. Here, we consider a linear model (the kernel method) to perform a supervised learning task (see SM). For our randomly generated $\mathbf{x}$, we assign them with one-dimensional outputs $y(\mathbf{x})$. The residual training error $\varepsilon(\mathbf{x})=z(\mathbf{x})-y(\mathbf{x})$, where $z(\mathbf{x})$ is the kernel method prediction, enters in the formula of the mean-square loss function $\mathcal{L}=\frac{1}{2} \sum_{\mathbf{x}} \varepsilon^{2}(\mathbf{x})$. We use the gradient descent algorithm to minimize $\mathcal{L}$. In the kernel method, the optimization process is exactly solvable, and the decay of the residual training error is exponential when the learning rate is small, where the decay rate can be predicted by kernel eigenvalues (see SM).

In Fig. 3, we perform different gradient descent processes for increasing $K_{\text {err }}$. We find that the increasing Kerr non-linearity will generically accelerate the gradient descent dynamics, consistent with our findings of kernel eigenspectra.


FIG. 3. We perform 30 different gradient descent processes, increasing $K_{\text {err }}$ from 0 to $\mathcal{O}\left(10^{3}\right) \mathrm{MHz} \times 2 \pi$ with a logarithmic scale and a fixed learning rate $10^{-3}$. The plots show the relative residual training error $|\varepsilon(t) / \varepsilon(0)|$ depending on the iteration step $t$ from 0 towards 500 , in the eigenvector direction of $K$ with the largest kernel eigenvalue. The theory (see SM) shows that this plot is independent of the choices of the supervised learning label $y$.

Generalization Error.-Moreover, one could quantify the performance of the machine learning models. As an example, we split half of the data as the training set, and we evaluate the loss function on the remaining half as the test set after we train the model. We consider learning a function $y(\mathbf{x})=\sum_{i=1}^{3} \sin ^{2}\left(\mathbf{x}_{i}^{2}\right)$ instead, where our time $T$ is rescaled to be $\mathcal{O}(1)$ and other components in $T$ are set by dimensional analysis accordingly. In Figure 4. we find that generalization error has non-trivial behavior for growing Kerr non-linearity, and for large Kerr values in the non-perturbative regime, it decays significantly. According to Refs. 38 42, generalization errors could be related to neural tangent kernel eigenvalues, and the kernel method is generalized well from good enough alignments. Thus, our finding provides good evidence that quantum Kerr learning will provide extra enhance-
ment in algorithm performances. See more detail in the SM.


FIG. 4. Evaluating the algorithm performance through the generalization error. We set the training set $\mathcal{T}$ to be the first half of the previous training set $\mathcal{A}$, and we denote the second half as the test set $\mathcal{B}$. We evaluate the generalization error using $\mathcal{L}_{\mathcal{B}}=\frac{1}{2|\mathcal{B}|} \sum_{\mathbf{x} \in \mathcal{B}} \varepsilon(\mathbf{x})^{2}$ for different Kerr coefficients.

Higher dimensions.-We can perform higherdimensional regression by using two quantum systems, instead of one, each encoding a subset of the data,

$$
\begin{align*}
\mathcal{K}= & \left\lvert\,\langle\psi \otimes \phi| \overline{\mathbb{T}} \exp \left[\frac{i}{\hbar} \int_{0}^{T^{\prime}} H\left(t^{\prime}\right) d t^{\prime}\right]\right.  \tag{3}\\
& \times\left.\mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} H(t) d t\right]|\psi \otimes \phi\rangle\right|^{2}
\end{align*}
$$

where we have assumed that we initially start in a product state. $H(t)$ could, again, be a general Hamiltonian. If $H(t)$ includes terms that couple the two subsystems, $|\psi\rangle$ and $|\phi\rangle$, this could potentially create entanglement between the two systems, increasing the complexity and making it harder for a classical computer to compute. It is also interesting to look at the case when $H(t)$ does not couple the two subsystems. For example, the Hamiltonian could represent two uncoupled Kerr resonators as

$$
\begin{equation*}
H(t)=H \otimes I+I \otimes H \tag{4}
\end{equation*}
$$

where $H$ is the Hamiltonian described in eq. (2) and $I$ is identity. In this initially unentangled and uncoupled example, our kernel reduces to a Hadamard product of the two uncoupled kernels,

$$
\begin{equation*}
\mathcal{K}=K_{\alpha} \circ K_{\beta}, \tag{5}
\end{equation*}
$$

where $K_{\alpha}\left(K_{\beta}\right)$ is the single-system Kerr kernel discussed in previous sections, and the subscript $(\alpha, \beta)$ represents two independent subsystems. Because kernels are symmetric, positive definite matrices, we can bound the properties of the eigenvalues of $\mathcal{K}$ using the spectra of $K$. For example, we have the following bound on the spectral radius, $\rho$, of the product kernel 43]

$$
\begin{equation*}
\rho(\mathcal{K}) \leq \rho\left(K_{\alpha}\right) \rho\left(K_{\beta}\right) \tag{6}
\end{equation*}
$$

This bound is known to be rather loose, and tighter bounds have been derived [43. As shown in Fig. 1, the maximum kernel eigenvalues of the single system grow with increasing $K_{\text {err }}$. For multiple unentangled and uncoupled systems, we can expect that the maximal eigenvalues of this product kernel grow faster, leading to increased performance in higher-dimensional data sets compared with the non-Kerr kernel. This can be extended to the product of many kernels, allowing for the learning of arbitrary dimensional data with collections of single oscillators. It is likely that the addition of entanglement, through a Hamiltonian term that couples the various systems, will further increase the performance [11].

Conclusion and outlooks.-Our paper opens up a novel direction by exploring the potential of quantum machine learning through circuit QED devices with non-trivial Kerr non-linearity. We find theoretical and numerical evidence where non-trivial Kerr coupling could significantly enhance the performance of the quantum kernel method, based on solid evaluations of kernel statistics and numerical optimization experiments. We strengthen our claims by applying the neural tangent kernel theory in machine learning, arguments from the theory of quantum complexity, and generalizations towards higher dimensions. Here, we suggest the following directions for future research.
a. Experimental implementation. It will be interesting to implement the proposal in our paper in the laboratory. When working with actual hardware, measurement noise and errors need to be considered. A combined design between theory and experiments could be obtained based on the theory of the kernel method, including trade-offs among the learning rate, the experimental precision for estimating the residual training error $\varepsilon$, and the number of gradient descent steps we could perform in the laboratory.
b. Extensions to multiple modes. One could generalize the above work to multiple modes, where we expect the coupling between different bosonic modes will strengthen the complexity of the kernel. It will be interesting to see if our approach will realize the power of multimode devices towards hard problems in quantum machine learning.
c. Theoretical considerations. It will be interesting to explore further the complexity foundations of our claims. The argument about the Kerr non-linearity and complexity, although promising, is not proven rigorously. A more solid statement about complexity might deepen our understanding of the algorithmic potential of the Kerr nonlinearity.

We thank David Meltzer, Daniel A. Roberts and Quntao Zhuang for useful discussions. We thank Liang Jiang for his numerous helpful suggestions. Work performed at the Center for Nanoscale Materials, a U.S. Department of Energy Office of Science User Facility, was supported by the U.S. DOE, Office of Basic Energy

Sciences, under Contract No. DE-AC02-06CH11357. JL is supported in part by International Business Machines (IBM) Quantum through the Chicago Quantum Exchange, and the Pritzker School of Molecular Engineering at the University of Chicago through AFOSR MURI (FA9550-21-1-0209). JL also serves as a scientific advisor of qBraid Co.. CZ and JY acknowledges support from the ARO (W911NF-18-1-0020, W911NF-18-10212), ARO MURI (W911NF-16-1-0349), AFOSR MURI (FA9550-19-1-0399, FA9550-21-1-0209), DoE Q-NEXT, NSF (EFMA-1640959, OMA-1936118, EEC-1941583), NTT Research, and the Packard Foundation (201339273).

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# Supplementary Material: Quantum Kerr Learning 

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## I. KERNEL METHOD AND THE GRADIENT DESCENT DYNAMICS

Here we provide a self-contained introduction to the basic methodology we use. For more details about kernel methods and neural tangent kernels, see [1-3].

We consider a simple supportive vector machine model. We start from a series of data ( $x_{\tilde{\alpha}}, y_{\tilde{\alpha}}$ ), where $\tilde{\alpha}$ denotes the index of data from the training set $\mathcal{A}$, and we are in a supervised learning task with the output label $y_{\tilde{\alpha}}$. We wish to fit the answer with the linear model,

$$
\begin{equation*}
z_{\delta}=z\left(x_{\delta}\right)=\sum_{\tilde{\alpha} \in \mathcal{A}} \theta_{\tilde{\alpha}} K\left(x_{\tilde{\alpha}}, x_{\delta}\right) \tag{S1}
\end{equation*}
$$

where $K$ is the kernel, $x_{\delta}$ is a general input and $\delta$ denotes the index of data from the whole input data set $\mathcal{D}$ (so we have $\mathcal{A} \subset \mathcal{D}$ ), and $\theta_{\delta}$ is a trainable variable. In the quantum kernel method [4], $K$ is evaluated using the quantum measurement. We set the loss function

$$
\begin{equation*}
L_{\mathcal{A}}=\frac{1}{2} \sum_{\tilde{\alpha} \in \mathcal{A}}\left(z_{\tilde{\alpha}}-y_{\tilde{\alpha}}\right)^{2}=\frac{1}{2} \sum_{\tilde{\alpha} \in \mathcal{A}} \varepsilon_{\tilde{\alpha}}^{2} \tag{S2}
\end{equation*}
$$

Here, $\varepsilon$ is the residual training error. We consider the simplest gradient descent dynamics,

$$
\begin{equation*}
\delta \theta_{\tilde{\alpha}}=\theta_{\tilde{\alpha}}(t+1)-\theta_{\tilde{\alpha}}(t)=-\eta \frac{d L_{\mathcal{A}}}{d \theta_{\tilde{\alpha}}}=-\eta \sum_{\tilde{\beta} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\beta}}}{d \theta_{\tilde{\alpha}}} \varepsilon_{\tilde{\beta}} \tag{S3}
\end{equation*}
$$

Moreover, since our model is linear, we have

$$
\begin{equation*}
\delta \varepsilon_{\tilde{\alpha}}=\sum_{\tilde{\beta} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}} \delta \theta_{\tilde{\beta}}=-\eta \sum_{\tilde{\beta}, \tilde{\gamma} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}} \frac{d \varepsilon_{\tilde{\gamma}}}{d \theta_{\tilde{\beta}}} \varepsilon_{\tilde{\gamma}} . \tag{S4}
\end{equation*}
$$

[^0]We define

$$
\begin{equation*}
K_{H, \tilde{\alpha} \tilde{\gamma}}=\sum_{\tilde{\beta} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}} \frac{d \varepsilon_{\tilde{\gamma}}}{d \theta_{\tilde{\beta}}} \tag{S5}
\end{equation*}
$$

So we get

$$
\begin{equation*}
\delta \varepsilon_{\tilde{\alpha}}=\sum_{\tilde{\beta} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}} \delta \theta_{\tilde{\beta}}=-\eta \sum_{\tilde{\gamma} \in \mathcal{A}} K_{H, \tilde{\alpha} \tilde{\gamma}} \varepsilon \tilde{\gamma} \tag{S6}
\end{equation*}
$$

The quantity $K_{H}$ is called the neural tangent kernel (NTK) [3, 5-11]. In the supportive vector machine case, $K_{H}$ is a constant,

$$
\begin{align*}
& \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}}=K\left(x_{\tilde{\beta}}, x_{\tilde{\alpha}}\right) \\
& K_{H, \tilde{\alpha} \tilde{\gamma}}=\sum_{\tilde{\beta} \in \mathcal{A}} \frac{d \varepsilon_{\tilde{\alpha}}}{d \theta_{\tilde{\beta}}} \frac{d \varepsilon_{\tilde{\gamma}}}{d \theta_{\tilde{\beta}}}=\sum_{\tilde{\beta} \in \mathcal{A}} K\left(x_{\tilde{\beta}}, x_{\tilde{\alpha}}\right) K\left(x_{\tilde{\beta}}, x_{\tilde{\gamma}}\right)=\sum_{\tilde{\beta} \in \mathcal{A}} K\left(x_{\tilde{\alpha}}, x_{\tilde{\beta}}\right) K\left(x_{\tilde{\beta}}, x_{\tilde{\gamma}}\right) \tag{S7}
\end{align*}
$$

Here, we have used the fact that $K$ is symmetric. If we use the compact matrix notation in the sample space, we have

$$
\begin{equation*}
K_{H}=K^{2} \tag{S8}
\end{equation*}
$$

The solution is analytically solved since $K_{H}$ or $K$ are constants,

$$
\begin{equation*}
\varepsilon_{\tilde{\alpha}}(t)=\left(\left(1-\eta K_{H}\right)^{t}\right)_{\tilde{\alpha} \tilde{\beta}} \varepsilon_{\tilde{\beta}}(0) \tag{S9}
\end{equation*}
$$

The properties of matrix $H=K^{2}$ would affect the efficiency and performance of the gradient descent. Say that we diagonalize the matrix $K$ by

$$
\begin{equation*}
K_{\tilde{\alpha} \tilde{\beta}}=\left(V \mathbb{K} V^{T}\right)_{\tilde{\alpha} \tilde{\beta}} \tag{S10}
\end{equation*}
$$

Here $V$ is the matrix of eigenvectors of $K$, and $\mathbb{K}$ is the matrix $K$ in the eigenspace,

$$
\begin{align*}
& \mathbb{K}_{=}=\operatorname{diag}\left(\kappa_{\tilde{\alpha}}\right) \\
& \mathbb{K}_{\tilde{\alpha} \tilde{\beta}}=\kappa_{\tilde{\alpha}} \delta_{\tilde{\alpha} \tilde{\beta}} \tag{S11}
\end{align*}
$$

We have

$$
\begin{equation*}
\varepsilon_{\tilde{\alpha}}(t)=\left(V\left(1-\eta \mathbb{K}^{2}\right)^{t} V^{T}\right)_{\tilde{\alpha} \tilde{\beta}} \varepsilon_{\tilde{\beta}}(0) \tag{S12}
\end{equation*}
$$

We define

$$
\begin{equation*}
u_{\tilde{\alpha}}=V_{\tilde{\alpha} \tilde{\beta}}^{T} \varepsilon_{\tilde{\beta}} \tag{S13}
\end{equation*}
$$

so we get

$$
\begin{equation*}
u_{\tilde{\alpha}}(t)=\left(1-\eta \kappa_{\tilde{\alpha}}^{2}\right)^{t} u_{\tilde{\alpha}}(0) . \tag{S14}
\end{equation*}
$$

Thus the spectrum of $K$ gives the characteristic decaying time of the residual training error in the inertial frame, $u$. More precisely, in the small $\eta$ limit, we get

$$
\begin{equation*}
u_{\tilde{\alpha}}(t) \approx e^{-\eta \kappa_{\tilde{\alpha}}^{2} t} u_{\tilde{\alpha}}(0) \tag{S15}
\end{equation*}
$$

Thus, a larger absolute value of $\kappa_{\tilde{\alpha}}$ will lead to a faster decay rate. If our kernel is Gaussian, there will be a sufficiently large amount of eigenvalues accumulated around zero, because of the long tails of the distribution. Thus, if our spectra of $K_{H}$ or $K$ are flatter, our performance is expected to be better.

Finally, we make a short discussion about the generalization error. For a general input data $\mathbf{x}_{\delta}$, the output at the end of training is given by [2, 3],

$$
\begin{equation*}
z_{\delta}(\infty)=z_{\delta}(0)-\sum_{\tilde{\alpha}, \tilde{\alpha}_{1} \in \mathcal{A}} H_{\delta \tilde{\alpha}} \tilde{H}^{\tilde{\alpha} \tilde{\alpha}_{1}} \varepsilon_{\tilde{\alpha}_{1}}(0) \tag{S16}
\end{equation*}
$$

Here, we use the notation where $\tilde{H}^{\tilde{\alpha}_{1} \tilde{\alpha}_{2}}=(H)_{\tilde{\alpha}_{1} \tilde{\alpha}_{2}}^{-1}$, the inverse of the matrix $H$ restricted at the subspace $\mathcal{A} \subset \mathcal{D}$. For example, in the main text in practice we take the initial trainable weights $W(0)=0$, so we get

$$
\begin{equation*}
z_{\delta}(\infty)=\sum_{\tilde{\alpha}, \tilde{\alpha}_{1} \in \mathcal{A}} H_{\delta \tilde{\alpha}} \tilde{H}^{\tilde{\alpha} \tilde{\alpha}_{1}} y_{\tilde{\alpha}_{1}} \tag{S17}
\end{equation*}
$$

Specifically, if we take a data point inside the space $\mathcal{A}$, we get

$$
\begin{equation*}
z_{\tilde{\delta}}(\infty)=\sum_{\tilde{\alpha}, \tilde{\alpha}_{1} \in \mathcal{A}} H_{\tilde{\delta} \tilde{\alpha}} \tilde{H}^{\tilde{\alpha} \tilde{\alpha}_{1}} y_{\tilde{\alpha}_{1}}=\sum_{\tilde{\alpha}_{1} \in \mathcal{A}} \delta_{\tilde{\delta}}^{\tilde{\alpha}_{1}} y_{\tilde{\alpha}_{1}}=y_{\tilde{\delta}} \tag{S18}
\end{equation*}
$$

by the definition of matrix inverse. One could define the generalization error as

$$
\begin{equation*}
\mathcal{L}_{\mathcal{B}}=\frac{1}{2|\mathcal{B}|} \sum_{\beta \in \mathcal{B}}\left(z_{\beta}-y_{\beta}\right)^{2}=\frac{1}{2|\mathcal{B}|} \sum_{\beta \in \mathcal{B}} \varepsilon_{\beta}^{2} . \tag{S19}
\end{equation*}
$$

Here, $\mathcal{B} \subset \mathcal{D}$ is a test set and $\mathcal{B} \neq \mathcal{A}$.

## II. THE HAMILTONIAN AND THE PERTURBATION THEORY

## A. Problem setup

We deal with the Hamiltonian,

$$
\begin{equation*}
H=H_{0}+H_{t}+H_{I}=H_{f}+H_{I} \tag{S20}
\end{equation*}
$$

where

$$
\begin{align*}
\frac{H_{0}}{\hbar} & =\omega_{m}\left(b^{\dagger} b+\frac{1}{2}\right) \\
\frac{H_{t}}{\hbar} & =\Omega\left(b e^{-i \omega_{L} t}+b^{\dagger} e^{i \omega_{L} t}\right) \\
\frac{H_{I}}{\hbar} & =K_{\mathrm{err}} b^{\dagger} b^{\dagger} b b \tag{S21}
\end{align*}
$$

Now we wish to evaluate

$$
\begin{equation*}
\left.K=\left|\langle\psi|\left(\overline{\mathbb{T}} \exp \left[\frac{i}{\hbar} \int_{0}^{T} H\left(\Omega, \omega_{L}\right) d t\right] \times \mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T^{\prime}} H\left(\Omega^{\prime}, \omega_{L}^{\prime}\right) d t\right]\right)\right| \psi\right\rangle\left.\right|^{2} \tag{S22}
\end{equation*}
$$

Note that we wish $\omega_{m}$ to be fixed.

## B. Free theory

Now we use the rotation frame method to simplify the problem. For an arbitrary quantum system generated by

$$
\begin{equation*}
i \hbar \frac{d}{d t}|\phi(t)\rangle=H(t)|\phi(t)\rangle \tag{S23}
\end{equation*}
$$

We assume a rotation frame such that

$$
\begin{equation*}
|\psi(t)\rangle=U_{R}(t)|\phi(t)\rangle \tag{S24}
\end{equation*}
$$

We get

$$
\begin{equation*}
i \hbar \frac{d}{d t}|\psi(t)\rangle=H_{R}(t)|\psi(t)\rangle \tag{S25}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{R}(t)=i \hbar \dot{U}_{R}(t) U^{\dagger}(t)+U_{R}(t) H(t) U_{R}^{\dagger}(t) \tag{S26}
\end{equation*}
$$

$H_{R}$ is, in a sense, a rotated definition of the Hamiltonian. Without Kerr non-linearity, we could apply it to

$$
\begin{equation*}
H_{f}(t)=H_{0}+H_{t}(t)=\hbar \omega_{m}\left(b^{\dagger} b+\frac{1}{2}\right)+\hbar \Omega\left(b e^{-i \omega_{L} t}+b^{\dagger} e^{i \omega_{L} t}\right) \tag{S27}
\end{equation*}
$$

with

$$
\begin{equation*}
U_{R}(t)=e^{i \omega_{L} t b^{\dagger} b} \tag{S28}
\end{equation*}
$$

So we have

$$
\begin{align*}
& H_{R}(t)=i \hbar \dot{U}_{R}(t) U_{R}^{\dagger}(t)+U_{R}(t) H_{f}(t) U_{R}^{\dagger}(t) \\
& =\hbar\left(\omega_{m}-\omega_{L}\right)\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right) \tag{S29}
\end{align*}
$$

In fact, we find $H_{R}$ is static. Now, remember that our feature map is

$$
\begin{equation*}
|\phi(\mathbf{x})\rangle=\mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} H_{f} d t\right]|\psi\rangle \tag{S30}
\end{equation*}
$$

which is driven by the Hamiltonian,

$$
\begin{equation*}
\frac{d|\phi(\mathbf{x})\rangle}{d T}=-\frac{i}{\hbar} H_{f}(T)|\phi(\mathbf{x})\rangle \tag{S31}
\end{equation*}
$$

So we define

$$
\begin{equation*}
|\psi(\mathbf{x})\rangle=U_{R}(T)|\phi(\mathbf{x})\rangle \tag{S32}
\end{equation*}
$$

and we have

$$
\begin{equation*}
|\psi(\mathbf{x})\rangle=\mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} H_{R} d t\right]|\psi\rangle=\exp \left[-\frac{i}{\hbar} H_{R} T\right]|\psi\rangle \tag{S33}
\end{equation*}
$$

Basically, we just prove,

$$
\begin{equation*}
U_{R}(T) \mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} H_{f} d t\right]=\exp \left[-\frac{i}{\hbar} H_{R} T\right] \tag{S34}
\end{equation*}
$$

Now, we could write down the kernel formula as

$$
\begin{equation*}
\left.K\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=\left|\left\langle\phi(\mathbf{x}) \mid \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\right|^{2}=\left|\langle\psi(\mathbf{x})| U_{R}(\mathbf{x}) U_{R}^{\dagger}\left(\mathbf{x}^{\prime}\right)\right| \psi\left(\mathbf{x}^{\prime}\right)\right\rangle\left.\right|^{2} \tag{S35}
\end{equation*}
$$

Here we denote

$$
\begin{equation*}
U_{R}(\mathbf{x})=U_{R}(T)=\exp \left(i \omega_{L} T b^{\dagger} b\right) \tag{S36}
\end{equation*}
$$

Now we start to compute $\psi(\mathbf{x})$. We have

$$
\begin{align*}
& H_{R}=\hbar\left(\omega_{m}-\omega_{L}\right)\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right) \\
& =-\hbar \Delta_{L}\left(b^{\dagger} b\right)+\hbar \Omega\left(b+b^{\dagger}\right)+\frac{1}{2} \hbar \omega_{m} \\
& =\hbar Q_{L}\left(b^{\dagger} b\right)+\hbar \Omega\left(b+b^{\dagger}\right)+\frac{1}{2} \hbar \omega_{m} \tag{S37}
\end{align*}
$$

where we define $\Delta_{L}=\omega_{L}-\omega_{m}=-Q_{L}$. We could perform the transformation,

$$
\begin{align*}
& b=a-\lambda \\
& b^{\dagger}=a^{\dagger}-\lambda \tag{S38}
\end{align*}
$$

where we assume that $\lambda$ is real. In this case we have

$$
\begin{align*}
& b^{\dagger} b=a^{\dagger} a-\lambda\left(a^{\dagger}+a\right)+\lambda^{2} \\
& =a^{\dagger} a-\lambda\left(b^{\dagger}+b+2 \lambda\right)+\lambda^{2} \\
& =a^{\dagger} a-\lambda\left(b^{\dagger}+b\right)-\lambda^{2} . \tag{S39}
\end{align*}
$$

We notice that we have $\left[a, a^{\dagger}\right]=1$. And we have

$$
\begin{align*}
& H_{R}=\hbar Q_{L}\left(b^{\dagger} b\right)+\hbar \Omega\left(b+b^{\dagger}\right)+\frac{1}{2} \hbar \omega_{m} \\
& =\hbar Q_{L} a^{\dagger} a-\lambda \hbar Q_{L}\left(b+b^{\dagger}\right)-\hbar Q_{L} \lambda^{2}+\hbar \Omega\left(b+b^{\dagger}\right)+\frac{1}{2} \hbar \omega_{m} \tag{S40}
\end{align*}
$$

We set

$$
\begin{equation*}
\lambda Q_{L}=\Omega \tag{S41}
\end{equation*}
$$

So we have

$$
\begin{equation*}
H_{R}=\hbar Q_{L} a^{\dagger} a-\hbar Q_{L} \lambda^{2}+\frac{1}{2} \hbar \omega_{m} \tag{S42}
\end{equation*}
$$

For simplicity, we assume that the initial state is expanded in the coherent state basis,

$$
\begin{equation*}
|\psi\rangle=\sum_{\alpha} f(\alpha)|\alpha\rangle_{b} \tag{S43}
\end{equation*}
$$

where $|\alpha\rangle_{b}$ is the coherent state corresponding to the creation operator $b^{\dagger}$. Note that

$$
\begin{equation*}
|\alpha\rangle_{b} \equiv\left|\alpha^{\prime}\right\rangle_{a}=|\alpha+\lambda\rangle_{a} \tag{S44}
\end{equation*}
$$

since

$$
\begin{align*}
& b|\alpha\rangle_{b}=(a-\lambda)|\alpha\rangle_{b}=\left(\alpha^{\prime}-\lambda\right)|\alpha\rangle_{b}=\alpha|\alpha\rangle_{b} \\
& a|\alpha\rangle_{b}=a\left|\alpha^{\prime}\right\rangle_{a}=\alpha^{\prime}\left|\alpha^{\prime}\right\rangle_{a}=(\alpha+\lambda)|\alpha\rangle_{b} \tag{S45}
\end{align*}
$$

The above equation is not precise, since the equation should be up to a global factor. A more precise calculation shows,

$$
\begin{align*}
& |\alpha+\lambda\rangle_{a}=e^{(\alpha+\lambda) a^{\dagger}-\left(\alpha^{*}+\lambda\right) a}|0\rangle_{a}=e^{(\alpha+\lambda) a^{\dagger}-\left(\alpha^{*}+\lambda\right) a} e^{-\lambda b^{\dagger}+\lambda b}|0\rangle_{b} \\
& =e^{(\alpha+\lambda)\left(b^{\dagger}+\lambda\right)-\left(\alpha^{*}+\lambda\right)(b+\lambda)} e^{-\lambda b^{\dagger}+\lambda b}|0\rangle_{b} \tag{S46}
\end{align*}
$$

since,

$$
\begin{align*}
& {\left[(\alpha+\lambda)\left(b^{\dagger}+\lambda\right)-\left(\alpha^{*}+\lambda\right)(b+\lambda),-\lambda b^{\dagger}+\lambda b\right]} \\
& =-\lambda\left(\alpha-\alpha^{*}\right) \tag{S47}
\end{align*}
$$

Using the BCH formula, we get

$$
\begin{equation*}
|\alpha+\lambda\rangle_{a}=e^{(i \operatorname{Im} \alpha) \lambda}|\alpha\rangle_{b} \tag{S48}
\end{equation*}
$$

Thus,

$$
\begin{align*}
& |\alpha+\lambda\rangle_{a}=e^{(i \operatorname{Im} \alpha) \lambda}|\alpha\rangle_{b} \\
& |\alpha\rangle_{a}=e^{(i \operatorname{Im} \alpha) \lambda}|\alpha-\lambda\rangle_{b} \tag{S49}
\end{align*}
$$

So we have

$$
\begin{equation*}
|\psi\rangle=\sum_{\alpha} f(\alpha)|\alpha\rangle_{b}=\sum_{\alpha} f(\alpha) e^{-(i \operatorname{Im} \alpha) \lambda}|\alpha+\lambda\rangle_{a}=\sum_{\alpha} f(\alpha-\lambda) e^{-(i \operatorname{Im} \alpha) \lambda}|\alpha\rangle_{a} \tag{S50}
\end{equation*}
$$

Thus, we get

$$
\begin{align*}
& \exp \left[-\frac{i}{\hbar} H_{R} T\right]|\psi\rangle=\exp \left[-\frac{i T}{\hbar}\left(\hbar Q_{L} a^{\dagger} a-\hbar Q_{L} \lambda^{2}+\frac{1}{2} \hbar \omega_{m}\right)\right]|\psi\rangle \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T} \sum_{\alpha^{\prime}} f(\alpha-\lambda) e^{-(i \operatorname{Im} \alpha) \lambda} \exp \left[-i Q_{L} T\left(a^{\dagger} a\right)\right]|\alpha\rangle_{a} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T} \sum_{\alpha} f(\alpha-\lambda) e^{-(i \operatorname{Im} \alpha) \lambda}\left|\alpha e^{-i Q_{L} T}\right\rangle_{a} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T} \sum_{\alpha} f(\alpha-\lambda) e^{i \operatorname{Im}\left(\alpha\left(e^{-i Q_{L} T}-1\right)\right) \lambda}\left|\alpha e^{-i Q_{L} T}-\lambda\right\rangle_{b} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T} \sum_{\beta} f\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[(\beta+\lambda)\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} \sum_{\beta} f\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \tag{S51}
\end{align*}
$$

So we have

$$
\begin{align*}
& U_{R}^{\dagger}|\psi(\mathbf{x})\rangle \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} \sum_{\beta} f\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} e^{-i \omega_{L} T b^{\dagger} b}|\beta\rangle_{b} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} \sum_{\beta} f\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}\left|e^{-i \omega_{L} T} \beta\right\rangle_{b} \\
& =e^{-\frac{1}{2} i \omega_{m} T+i Q_{L} \lambda^{2} T-i \lambda^{2} \sin \left(Q_{L} T\right)} \sum_{\beta} f\left(\left(e^{i \omega_{L} T} \beta+\lambda\right) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta e^{i \omega_{L} T}\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \tag{S52}
\end{align*}
$$

We could re-write it by

$$
\begin{align*}
& \left(e^{i \omega_{L} T} \beta+\lambda\right) e^{i Q_{L} T}-\lambda=e^{i \omega_{m} T} \beta-\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} \\
& \omega_{L}=\omega_{m}-Q_{L} \tag{S53}
\end{align*}
$$

Now we visit a specific situation, where

$$
\begin{equation*}
|\psi\rangle=\sum_{\alpha} f(\alpha)|\alpha\rangle_{b} \Rightarrow \int d \alpha f(\alpha)|\alpha\rangle_{b}=\int d \alpha \delta\left(\alpha-\alpha_{i}\right)|\alpha\rangle_{b}=\left|\alpha_{i}\right\rangle_{b} \tag{S54}
\end{equation*}
$$

and we visit the case where $\alpha_{i}$ is real. So we impose

$$
\begin{equation*}
e^{i \omega_{m} T} \beta-\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}}=\alpha_{i} \tag{S55}
\end{equation*}
$$

And we have

$$
\begin{equation*}
\beta=e^{-i \omega_{m} T} \alpha_{i}+\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} e^{-i \omega_{m} T} \tag{S56}
\end{equation*}
$$

So we get

$$
\begin{align*}
& U_{R}^{\dagger}|\psi(\mathbf{x})\rangle=e^{-\frac{1}{2} i \omega_{m} T+i \frac{\Omega^{2}}{Q_{L}} T-i \frac{\Omega\left(\alpha_{i} Q_{L}+\Omega\right)}{Q_{L}^{2}} \sin \left(Q_{L} T\right)}\left|e^{-i \omega_{m} T} \alpha_{i}+\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} e^{-i \omega_{m} T}\right\rangle_{b} \\
& \propto\left|e^{-i \omega_{m} T} \alpha_{i}+\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} e^{-i \omega_{m} T}\right\rangle_{b} \tag{S57}
\end{align*}
$$

## C. Perturbation theory

We could use a similar trick to study the perturbation theory of the Kerr non-linearity. First, we notice that

$$
\begin{gather*}
H_{I}=\hbar K_{\mathrm{er}} b^{\dagger} b^{\dagger} b b=\hbar K_{\mathrm{err}} b^{\dagger} b b^{\dagger} b+\hbar K_{\mathrm{err}} b^{\dagger}\left[b^{\dagger}, b\right] b \\
=\hbar K_{\mathrm{err}} b^{\dagger} b b^{\dagger} b-\hbar K_{\mathrm{err}} b^{\dagger} b=\hbar K_{\mathrm{err}} N^{2}-\hbar K_{\mathrm{err}} N \tag{S58}
\end{gather*}
$$

So we have

$$
\begin{equation*}
H_{R}(t)=H_{R}=\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar K_{\mathrm{err}} b^{\dagger} b^{\dagger} b b+\hbar \Omega\left(b+b^{\dagger}\right) \tag{S59}
\end{equation*}
$$

Moreover, using the Feynman's disentangling formula,

$$
\begin{equation*}
\exp (T(A+B))=\exp (T A) \mathbb{T} \exp \left[\int_{0}^{T} \tilde{B}(\tau) d \tau\right] \tag{S60}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{B}(\tau) \equiv \exp (-\tau A) B \exp (\tau A) \tag{S61}
\end{equation*}
$$

So we get

$$
\begin{align*}
& \exp \left[-\frac{i}{\hbar} H_{R} T\right]|\psi\rangle \\
& =\exp \left[-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) T\right] \mathbb{T} \exp \left[-\frac{i}{\hbar} \int_{0}^{T} \tilde{H}_{I}(\tau) d \tau\right]|\psi\rangle \\
& =e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) T}\left(1-\frac{i}{\hbar} \int_{0}^{T} \tilde{H}_{I}(\tau) d \tau+\mathcal{O}\left(K_{\text {err }}^{2}\right)\right)|\psi\rangle \tag{S62}
\end{align*}
$$

where

$$
\begin{equation*}
\tilde{H}_{I}(\tau)=\hbar K_{\mathrm{err}} e^{\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} \tag{S63}
\end{equation*}
$$

We could write

$$
\begin{equation*}
|\psi(\mathbf{x})\rangle=\exp \left[-\frac{i}{\hbar} H_{R} T\right]|\psi\rangle=\left|\psi_{0}(\mathbf{x})\right\rangle+|\delta \psi(\mathbf{x})\rangle \tag{S64}
\end{equation*}
$$

where

$$
\begin{equation*}
\left|\psi_{0}(\mathbf{x})\right\rangle=e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) T}|\psi\rangle \tag{S65}
\end{equation*}
$$

which has been computed before, and

$$
\begin{align*}
& |\delta \psi(\mathbf{x})\rangle=-\frac{i}{\hbar} e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) T} \int_{0}^{T} \tilde{H}_{I}(\tau)|\psi\rangle d \tau=-i K_{\mathrm{err}} e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) T} \\
& \int_{0}^{T} e^{\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau}|\psi\rangle d \tau \tag{S66}
\end{align*}
$$

Moreover, we have

$$
\begin{align*}
& \tilde{H}_{I}(\tau)=\hbar K_{\operatorname{err}} e^{\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\frac{1}{2} \hbar \omega_{m}+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} \\
& =\hbar K_{\mathrm{err}} e^{\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L}\left(b^{\dagger} b\right)+\hbar \Omega\left(b+b^{\dagger}\right)\right) \tau} \\
& =\hbar K_{\mathrm{err}} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a-\hbar Q_{L} \lambda^{2}\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a-\hbar Q_{L} \lambda^{2}\right) \tau} \\
& =\hbar K_{\mathrm{err}} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} b b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \tag{S67}
\end{align*}
$$

We will use the identity:

$$
\begin{align*}
& b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}=(a-\lambda) e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}=a e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}-\lambda e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \\
& =e^{-i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} a-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda=e^{-i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}(b+\lambda)-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda \\
& =e^{-i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b+e^{-i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda \\
& =e^{-i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b+\left(e^{-i Q_{L} \tau}-1\right) e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda . \tag{S68}
\end{align*}
$$

And similarly

$$
\begin{align*}
& b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}=\left(a^{\dagger}-\lambda\right) e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}=a^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}-\lambda e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \\
& =e^{i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} a^{\dagger}-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda=e^{i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau}\left(b^{\dagger}+\lambda\right)-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda \\
& =e^{i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger}+e^{i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda-e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda \\
& =e^{i Q_{L} \tau} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger}+\left(e^{i Q_{L} \tau}-1\right) e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \lambda \tag{S69}
\end{align*}
$$

We have

$$
\begin{equation*}
\tilde{H}_{I}(\tau)=\hbar K_{\mathrm{err}} e^{-i Q_{L} \tau} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b+\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \tag{S70}
\end{equation*}
$$

Finally, we get

$$
\begin{align*}
& \tilde{H}_{I}(\tau)=\hbar K_{\mathrm{err}} b^{\dagger} b^{\dagger} b b \\
& +\lambda \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau} b^{\dagger} b b \\
& +\lambda \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right) e^{-2 i Q_{L} \tau} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b b \\
& +\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b \\
& +\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \tag{S71}
\end{align*}
$$

Moreover, similar techniques could be used to deal with terms with $\lambda$. We have

$$
\begin{align*}
& \lambda \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right) e^{-2 i Q_{L} \tau} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b b \\
& =\lambda \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau} b^{\dagger} b b+\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right)^{2} e^{-2 i Q_{L} \tau} b b \tag{S72}
\end{align*}
$$

and

$$
\begin{align*}
& \lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau} e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b \\
& =\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b \\
& +\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau}\left(e^{i Q_{L} \tau}-1\right) e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b \\
& =\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{i Q_{L} \tau} b^{\dagger} b^{\dagger} b \\
& +2 \lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right) b^{\dagger} b \\
& +\lambda^{3} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{-i Q_{L} \tau}\left(e^{i Q_{L} \tau}-1\right)^{2} b \tag{S73}
\end{align*}
$$

and

$$
\begin{align*}
& \lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} b^{\dagger} b^{\dagger} b e^{-\frac{i}{\hbar}\left(\hbar Q_{L} a^{\dagger} a\right) \tau} \\
& =\lambda \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right) e^{i Q_{L} \tau} b^{\dagger} b^{\dagger} b \\
& +\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right) b^{\dagger} b+\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right) b^{\dagger} b \\
& +\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2} e^{2 i Q_{L} \tau} b^{\dagger} b^{\dagger} \\
& +\lambda^{3} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right)^{2} e^{-i Q_{L} \tau} b \\
& +2 \lambda^{3} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2}\left(e^{i Q_{L} \tau}-1\right) e^{i Q_{L} \tau} b^{\dagger} \\
& +\lambda^{4} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2}\left(e^{i Q_{L} \tau}-1\right)^{2} \tag{S74}
\end{align*}
$$

Thus, collecting all the terms, we have

$$
\begin{align*}
& \mathcal{O}\left(\lambda^{0}\right): \hbar K_{\mathrm{err}} b^{\dagger} b^{\dagger} b b \\
& \mathcal{O}\left(\lambda^{1}\right): 2 \lambda \hbar K_{\mathrm{err}}\left(1-e^{-i Q_{L} \tau}\right) b^{\dagger} b b+2 \lambda \hbar K_{\mathrm{err}}\left(1-e^{i Q_{L} \tau}\right) b^{\dagger} b^{\dagger} b \\
& \mathcal{O}\left(\lambda^{2}\right): \lambda^{2} \hbar K_{\mathrm{err}}\left(e^{i Q_{L} \tau}-1\right)^{2} e^{-2 i Q_{L} \tau} b b+\lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2} e^{2 i Q_{L} \tau} b^{\dagger} b^{\dagger} \\
& +4 \lambda^{2} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right) b^{\dagger} b \\
& \mathcal{O}\left(\lambda^{3}\right): 2 \lambda^{3} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)\left(e^{i Q_{L} \tau}-1\right)^{2} e^{-i Q_{L} \tau} b \\
& +2 \lambda^{3} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2}\left(e^{i Q_{L} \tau}-1\right) e^{i Q_{L} \tau} b^{\dagger} \\
& \mathcal{O}\left(\lambda^{4}\right): \lambda^{4} \hbar K_{\mathrm{err}}\left(e^{-i Q_{L} \tau}-1\right)^{2}\left(e^{i Q_{L} \tau}-1\right)^{2} \tag{S75}
\end{align*}
$$

After reorganizing terms and performing integrals, we get

$$
\begin{equation*}
-\frac{i}{\hbar} \int_{0}^{T} \tilde{H}_{I}(\tau) d \tau|\psi\rangle=\sum_{\alpha} g_{0}(\alpha)|\alpha\rangle_{b}+\sum_{\alpha} g_{1}(\alpha) b^{\dagger}|\alpha\rangle_{b}+\sum_{\alpha} g_{2}(\alpha) b^{\dagger} b^{\dagger}|\alpha\rangle_{b} \tag{S76}
\end{equation*}
$$

where

$$
\begin{align*}
& g_{0}(\alpha)=K_{\mathrm{err}} \frac{\lambda^{2} e^{-2 i T Q_{L}}}{2 Q_{L}}\binom{(\alpha+\lambda)^{2}+e^{2 i T Q_{L}}\left(3 \alpha(\alpha+2 \lambda)-2 i T\left(\alpha^{2}+6 \alpha \lambda+6 \lambda^{2}\right) Q_{L}\right)}{+4 \lambda(\alpha+2 \lambda) e^{3 i T Q_{L}}-4(\alpha+\lambda)(\alpha+2 \lambda) e^{i T Q_{L}}+\lambda^{2}\left(-e^{4 i T Q_{L}}\right)} f(\alpha), \\
& g_{1}(\alpha)=K_{\mathrm{err}} \frac{\lambda e^{-i T Q_{L}}}{Q_{L}}\binom{-2(\alpha+\lambda)^{2}+e^{i T Q_{L}}\left(2 \alpha^{2}-3 \lambda^{2}-2 i T(\alpha+\lambda)(\alpha+3 \lambda) Q_{L}\right)}{+2 \lambda(2 \alpha+3 \lambda) e^{2 i T Q_{L}}+\lambda^{2}\left(-e^{3 i T Q_{L}}\right)} f(\alpha) \\
& g_{2}(\alpha)=\frac{K_{\mathrm{err}}}{2 Q_{L}}\left(\lambda\left(-1+e^{i T Q_{L}}\right)\left(4 \alpha-\lambda\left(-3+e^{i T Q_{L}}\right)\right)-2 i T(\alpha+\lambda)^{2} Q_{L}\right) f(\alpha) \tag{S77}
\end{align*}
$$

Moreover, we know that

$$
\begin{align*}
& |\delta \psi(\mathbf{x})\rangle=e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} T a^{\dagger} a} \sum_{\alpha} g_{0}(\alpha)|\alpha\rangle_{b}+e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} T a^{\dagger} a} \sum_{\alpha} g_{1}(\alpha) b^{\dagger}|\alpha\rangle_{b} \\
& +e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} T a^{\dagger} a} \sum_{\alpha} g_{2}(\alpha) b^{\dagger} b^{\dagger}|\alpha\rangle_{b} \tag{S78}
\end{align*}
$$

Since

$$
\begin{align*}
& e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} T a^{\dagger} a} \sum_{\alpha} g_{0}(\alpha)|\alpha\rangle_{b} \\
& =e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} \sum_{\beta} g_{0}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \tag{S79}
\end{align*}
$$

Furthermore, using the identity,

$$
\begin{equation*}
e^{-i Q_{L} T a^{\dagger} a} b^{\dagger}=e^{-i Q_{L} T} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}+\left(e^{-i Q_{L} T}-1\right) e^{-i Q_{L} T a^{\dagger} a} \lambda \tag{S80}
\end{equation*}
$$

we have

$$
\begin{align*}
& e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} a^{\dagger} a T} \sum_{\alpha} g_{1}(\alpha) b^{\dagger}|\alpha\rangle_{b} \\
& =e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} e^{-i Q_{L} T} \sum_{\alpha} g_{1}(\alpha) b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}|\alpha\rangle_{b} \\
& +\lambda e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)}\left(e^{-i Q_{L} T}-1\right) \sum_{\alpha} g_{1}(\alpha) e^{-i Q_{L} T a^{\dagger} a}|\alpha\rangle_{b} \\
& =e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{-i Q_{L} T} \sum_{\beta} g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} b^{\dagger}|\beta\rangle_{b} \\
& +\lambda e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]}\left(e^{-i Q_{L} T}-1\right) \sum_{\beta} g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \tag{S81}
\end{align*}
$$

and we have

$$
\begin{align*}
& e^{-i Q_{L} T a^{\dagger} a} b^{\dagger} b^{\dagger}=e^{-i Q_{L} T} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a} b^{\dagger}+\lambda\left(e^{-i Q_{L} T}-1\right) e^{-i Q_{L} T a^{\dagger} a} b^{\dagger} \\
& =e^{-i Q_{L} T} b^{\dagger}\left(e^{-i Q_{L} T} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}+\left(e^{-i Q_{L} T}-1\right) e^{-i Q_{L} T a^{\dagger} a} \lambda\right) \\
& +\lambda\left(e^{-i Q_{L} T}-1\right)\left(e^{-i Q_{L} T} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}+\left(e^{-i Q_{L} T}-1\right) e^{-i Q_{L} T a^{\dagger} a} \lambda\right) \\
& =e^{-2 i Q_{L} T} b^{\dagger} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}+2 \lambda\left(e^{-2 i Q_{L} T}-e^{-i Q_{L} T}\right) b^{\dagger} e^{-i Q_{L} T a^{\dagger} a} \\
& +\lambda^{2}\left(e^{-2 i Q_{L} T}-2 e^{-i Q_{L} T}+1\right) e^{-i Q_{L} T a^{\dagger} a} \tag{S82}
\end{align*}
$$

So we have

$$
\begin{align*}
& \left.e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right.}\right) e^{-i Q_{L} T a^{\dagger} a} \sum_{\alpha} g_{2}(\alpha) b^{\dagger} b^{\dagger}|\alpha\rangle_{b} \\
& \left.=e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right.}\right) e^{-2 i Q_{L} T} \sum_{\alpha} g_{2}(\alpha) b^{\dagger} b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}|\alpha\rangle_{b} \\
& +e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} 2 \lambda\left(e^{-2 i Q_{L} T}-e^{-i Q_{L} T}\right) \sum_{\alpha} g_{2}(\alpha) b^{\dagger} e^{-i Q_{L} T a^{\dagger} a}|\alpha\rangle_{b} \\
& +e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)} \lambda^{2}\left(e^{-2 i Q_{L} T}-2 e^{-i Q_{L} T}+1\right) \sum_{\alpha} g_{2}(\alpha) e^{-i Q_{L} T a^{\dagger} a}|\alpha\rangle_{b} \\
& =e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{-2 i Q_{L} T} \\
& \sum_{\beta} g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} b^{\dagger} b^{\dagger}|\beta\rangle_{b} \\
& +e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} 2 \lambda\left(e^{-2 i Q_{L} T}-e^{-i Q_{L} T}\right) \\
& \sum_{\beta} g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} b^{\dagger}|\beta\rangle_{b} \\
& +e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} \lambda^{2}\left(e^{-2 i Q_{L} T}-2 e^{-i Q_{L} T}+1\right) \\
& \sum_{\beta} g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}|\beta\rangle_{b} \tag{S83}
\end{align*}
$$

Thus, we could define

$$
\begin{equation*}
|\delta \psi(\mathbf{x})\rangle=\sum_{\beta} h_{0}(\beta)|\beta\rangle_{b}+\sum_{\beta} h_{1}(\beta) b^{\dagger}|\beta\rangle_{b}+\sum_{\beta} h_{2}(\beta) b^{\dagger} b^{\dagger}|\beta\rangle_{b} \tag{S84}
\end{equation*}
$$

where

$$
\begin{align*}
& h_{0}(\beta)=e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} g_{0}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& +\lambda e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}\left(e^{-i Q_{L} T}-1\right) g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& +\lambda^{2} e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}\left(e^{-2 i Q_{L} T}-2 e^{-i Q_{L} T}+1\right) g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& h_{1}(\beta)=e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} e^{-i Q_{L} T} g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& +2 \lambda e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]}\left(e^{-2 i Q_{L} T}-e^{-i Q_{L} T}\right) g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& h_{2}(\beta)=e^{i T\left(\frac{\Omega^{2}}{\omega_{m}-\omega_{L}}-\frac{1}{2} \omega_{m}\right)+i \lambda \operatorname{Im}\left[\lambda\left(1-e^{i Q_{L} T}\right)\right]} e^{i \lambda \operatorname{Im}\left[\beta\left(1-e^{i Q_{L} T}\right)\right]} e^{-2 i Q_{L} T} g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \tag{S85}
\end{align*}
$$

Thus we have

$$
\begin{equation*}
|\delta \phi(\mathbf{x})\rangle=e^{-i \omega_{L} T b^{\dagger} b}|\delta \psi(\mathbf{x})\rangle=\sum_{\beta} s_{0}(\beta)|\beta\rangle_{b}+\sum_{\beta} s_{1}(\beta) b^{\dagger}|\beta\rangle_{b}+\sum_{\beta} s_{2}(\beta) b^{\dagger} b^{\dagger}|\beta\rangle_{b} \tag{S86}
\end{equation*}
$$

where

$$
\begin{align*}
& s_{0}(\beta)=h_{0}\left(\beta e^{i \omega_{L} T}\right) \\
& s_{1}(\beta)=h_{1}\left(\beta e^{i \omega_{L} T}\right) e^{-i T \omega_{L}} \\
& s_{2}(\beta)=h_{2}\left(\beta e^{i \omega_{L} T}\right) e^{-2 i T \omega_{L}} \tag{S87}
\end{align*}
$$

Now, consider the kernel function,

$$
\begin{align*}
& K\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=\left|\left\langle\phi(\mathbf{x}) \mid \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\right|^{2} \\
& =\left|\left(\left\langle\phi_{0}(\mathbf{x})\right|+\langle\delta \phi(\mathbf{x})|\right)\left(\left|\phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle+\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\right)\right|^{2} \\
& \left.=\left|\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle+\langle\delta \phi(\mathbf{x})|\right| \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle+\left\langle\phi_{0}(\mathbf{x})\right|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle+\left.\langle\delta \phi(\mathbf{x})|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\right|^{2} \\
& =\left(\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle+\langle\delta \phi(\mathbf{x})|\left|\phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle+\left\langle\phi_{0}(\mathbf{x})\right|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle+\langle\delta \phi(\mathbf{x})|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\right) \\
& \left(\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right) \mid \phi_{0}(\mathbf{x})\right\rangle+\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right)\right||\delta \phi(\mathbf{x})\rangle+\left\langle\delta \phi\left(\mathbf{x}^{\prime}\right)\right|\left|\phi_{0}(\mathbf{x})\right\rangle+\left\langle\delta \phi\left(\mathbf{x}^{\prime}\right)\right||\delta \phi(\mathbf{x})\rangle\right) \\
& =\left|\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\right|^{2}+\langle\delta \phi(\mathbf{x})|\left|\phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right) \mid \phi_{0}(\mathbf{x})\right\rangle \\
& +\left\langle\phi_{0}(\mathbf{x})\right|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right) \mid \phi_{0}(\mathbf{x})\right\rangle+\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right)\right||\delta \phi(\mathbf{x})\rangle \\
& +\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\delta \phi\left(\mathbf{x}^{\prime}\right)\right|\left|\phi_{0}(\mathbf{x})\right\rangle+\mathcal{O}\left(K_{\text {err }}^{2}\right) \\
& =\kappa_{0}\left(\mathbf{x}, \mathbf{x}^{\prime}\right)+\langle\delta \phi(\mathbf{x})|\left|\phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right) \mid \phi_{0}(\mathbf{x})\right\rangle \\
& +\left\langle\phi_{0}(\mathbf{x})\right|\left|\delta \phi\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right) \mid \phi_{0}(\mathbf{x})\right\rangle+\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\phi_{0}\left(\mathbf{x}^{\prime}\right)\right||\delta \phi(\mathbf{x})\rangle \\
& +\left\langle\phi_{0}(\mathbf{x}) \mid \phi_{0}\left(\mathbf{x}^{\prime}\right)\right\rangle\left\langle\delta \phi\left(\mathbf{x}^{\prime}\right)\right|\left|\phi_{0}(\mathbf{x})\right\rangle+\mathcal{O}\left(K_{\mathrm{err}}^{2}\right) . \tag{S88}
\end{align*}
$$

We only need to compute a single term, and other terms would follow the symmetric and hermitian operations. We know from the previous analysis

Finally we have

$$
\begin{equation*}
\left\langle\phi_{0}(\overline{\mathbf{x}}) \mid \delta \phi(\mathbf{x})\right\rangle=\sum_{\beta, \bar{\beta}} G(\beta, \bar{\beta} ; \mathbf{x}, \overline{\mathbf{x}}) e_{b}(\beta, \bar{\beta}) \tag{S90}
\end{equation*}
$$

where

$$
\begin{equation*}
e_{b}\left(\beta_{1}, \beta_{2}\right)=\left\langle\beta_{1} \mid \beta_{2}\right\rangle_{b}=e^{\beta_{1}^{*} \beta_{2}-\left|\beta_{1}\right|^{2} / 2-\left|\beta_{2}\right|^{2} / 2} \tag{S91}
\end{equation*}
$$

and

$$
\begin{align*}
& G(\beta, \bar{\beta} ; \mathbf{x}, \overline{\mathbf{x}})=e^{-i \bar{T}\left(\frac{\bar{\Omega}^{2}}{\omega_{m}-\bar{\omega}_{L}}-\frac{1}{2} \omega_{m}\right)} e^{i \frac{\bar{\Omega}^{2}}{\left(\omega_{m}-\bar{\omega}_{L}\right)^{2}} \sin \left(\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}\right)} e^{-i \frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)} \operatorname{Im}\left[\bar{\beta} e^{i \bar{\omega}_{L} \bar{T}}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right]} \\
& f^{*}\left(e^{\left.i \omega_{m} \bar{T} \bar{\beta}-\frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right) s_{0}(\mathbf{x}, \beta)}\right. \\
& +\bar{\beta} e^{-i \bar{T}\left(\frac{\bar{\Omega}^{2}}{\omega_{m}-\bar{\omega}_{L}}-\frac{1}{2} \omega_{m}\right)} e^{i \frac{\bar{\Omega}^{2}}{\left(\omega_{m}-\bar{\omega}_{L}\right)^{2}} \sin \left(\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}\right)} e^{-i \frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)} \operatorname{Im}\left[\bar{\beta}^{i \bar{\omega}_{L} \bar{T}}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right]} \\
& f^{*}\left(e^{\left.i \omega_{m} \bar{T} \bar{\beta}-\frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right) s_{1}(\mathbf{x}, \beta)}\right. \\
& +\bar{\beta}^{2} e^{-i \bar{T}\left(\frac{\bar{\Omega}^{2}}{\omega_{m}-\bar{\omega}_{L}}-\frac{1}{2} \omega_{m}\right)} e^{i \frac{\bar{\Omega}^{2}}{\left(\omega_{m}-\bar{\omega}_{L}\right)^{2}} \sin \left(\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}\right)} e^{-i \frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)} \operatorname{Im}\left[\bar{\beta}^{i \bar{\omega}_{L} \bar{T}}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right]} \\
& f^{*}\left(e^{\left.i \omega_{m} \bar{T} \bar{\beta}-\frac{\bar{\Omega}}{\left(\omega_{m}-\bar{\omega}_{L}\right)}\left(1-e^{i\left(\omega_{m}-\bar{\omega}_{L}\right) \bar{T}}\right)\right) s_{2}(\mathbf{x}, \beta)} .\right. \tag{S92}
\end{align*}
$$

Moreover, if we assume

$$
\begin{equation*}
\left\langle\phi_{0}(\overline{\mathbf{x}}) \mid \phi_{0}(\mathbf{x})\right\rangle=G_{0}(\mathbf{x}, \overline{\mathbf{x}}), \tag{S93}
\end{equation*}
$$

we get

$$
\begin{equation*}
K\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=K_{0}\left(\mathbf{x}, \mathbf{x}^{\prime}\right)+\delta K\left(\mathbf{x}, \mathbf{x}^{\prime}\right)+\mathcal{O}\left(K_{\mathrm{err}}^{2}\right) \tag{S94}
\end{equation*}
$$

and,

$$
\begin{align*}
& \delta K(\mathbf{x}, \overline{\mathbf{x}})=\left(\sum_{\beta, \bar{\beta}} G_{0}^{*}(\mathbf{x}, \overline{\mathbf{x}}) G(\beta, \bar{\beta} ; \mathbf{x}, \overline{\mathbf{x}}) e_{b}(\beta, \bar{\beta})\right)+\left(\sum_{\beta, \bar{\beta}} G_{0}^{*}(\mathbf{x}, \overline{\mathbf{x}}) G^{*}(\bar{\beta}, \beta ; \overline{\mathbf{x}}, \mathbf{x}) e_{b}(\beta, \bar{\beta})\right) \\
& +\left(\sum_{\beta, \bar{\beta}} G_{0}(\mathbf{x}, \overline{\mathbf{x}}) G^{*}(\beta, \bar{\beta} ; \mathbf{x}, \overline{\mathbf{x}}) e_{b}(\beta, \bar{\beta})\right)+\left(\sum_{\beta, \bar{\beta}} G_{0}(\mathbf{x}, \overline{\mathbf{x}}) G(\bar{\beta}, \beta ; \overline{\mathbf{x}}, \mathbf{x}) e_{b}(\beta, \bar{\beta})\right) . \tag{S95}
\end{align*}
$$

Moreover, we use the setup again where $\alpha=\alpha_{i}$. We have

$$
\begin{align*}
& \left|\phi_{0}(\mathbf{x})\right\rangle=e^{-\frac{1}{2} i \omega_{m} T+i \frac{\Omega^{2}}{Q_{L}} T-i \frac{\Omega\left(\alpha_{i} Q_{L}+\Omega\right)}{Q_{L}^{2}} \sin \left(Q_{L} T\right)}\left|e^{-i \omega_{m} T} \alpha_{i}+\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} e^{-i \omega_{m} T}\right\rangle_{b}, \\
& \left\langle\phi_{0}(\overline{\mathbf{x}})\right|=e^{\frac{1}{2} i \omega_{m} \bar{T}-i \frac{\bar{\Omega}^{2}}{Q_{L}} \bar{T}+i \frac{\bar{\Omega}\left(\alpha_{i} \bar{Q}_{L}+\bar{\Omega}\right)}{\bar{Q}_{L}^{2}} \sin \left(\bar{Q}_{L} \bar{T}\right)}\left\langle e^{-i \omega_{m} \bar{T}} \alpha_{i}+\left.\left(1-e^{i \bar{Q}_{L} \bar{T}}\right) \frac{\bar{\Omega}}{\bar{Q}_{L}} e^{-i \omega_{m} \bar{T}}\right|_{b}\right. \tag{S96}
\end{align*}
$$

Thus,

$$
\begin{equation*}
K_{0}(\mathbf{x}, \overline{\mathbf{x}})=\left|\left\langle\phi_{0}(\overline{\mathbf{x}}) \mid \phi_{0}(\mathbf{x})\right\rangle\right|^{2}=\left|e_{b}(\beta, \bar{\beta})\right|^{2} \tag{S97}
\end{equation*}
$$

where

$$
\begin{align*}
& \beta=e^{-i \omega_{m} T} \alpha_{i}+\left(1-e^{i Q_{L} T}\right) \frac{\Omega}{Q_{L}} e^{-i \omega_{m} T} \\
& \bar{\beta}=e^{-i \omega_{m} \bar{T}} \alpha_{i}+\left(1-e^{i \bar{Q}_{L} \bar{T}}\right) \frac{\bar{\Omega}}{\bar{Q}_{L}} e^{-i \omega_{m} \bar{T}} \tag{S98}
\end{align*}
$$

and we define

$$
\begin{align*}
& p(\mathbf{x})=e^{-\frac{1}{2} i \omega_{m} T+i \frac{\Omega^{2}}{Q_{L}} T-i \frac{\Omega\left(\alpha_{i} Q_{L}+\Omega\right)}{Q_{L}^{2}} \sin \left(Q_{L} T\right)} \\
& \left\langle\phi_{0}(\overline{\mathbf{x}}) \mid \phi_{0}(\mathbf{x})\right\rangle=p^{*}(\overline{\mathbf{x}}) p(\mathbf{x}) e_{b}(\beta, \bar{\beta}) \tag{S99}
\end{align*}
$$

Moreover, we have

$$
\delta K(\mathbf{x}, \overline{\mathbf{x}})=\left|e_{b}(\beta, \bar{\beta})\right|^{2}\left(\begin{array}{l}
p^{*}(\mathbf{x}) s_{0}(\mathbf{x}, \beta)  \tag{S100}\\
+p^{*}(\mathbf{x}) \bar{\beta} s_{1}(\mathbf{x}, \beta) \\
+p^{*}(\mathbf{x}) \bar{\beta}^{2} s_{2}(\mathbf{x}, \beta) \\
+ \text { c.c. } \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}) \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}, \text { c.c. })
\end{array}\right) .
$$

There is a further simplification for the phase factor. We could define

$$
\begin{align*}
& \tilde{h}_{0}(\beta)=g_{0}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right)+\lambda\left(e^{-i Q_{L} T}-1\right) g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& +\lambda^{2}\left(e^{-2 i Q_{L} T}-2 e^{-i Q_{L} T}+1\right) g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right) \\
& \tilde{h}_{1}(\beta)=e^{-i Q_{L} T} g_{1}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right)+2 \lambda\left(e^{-2 i Q_{L} T}-e^{-i Q_{L} T}\right) g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right), \\
& \tilde{h}_{2}(\beta)=e^{-2 i Q_{L} T} g_{2}\left((\beta+\lambda) e^{i Q_{L} T}-\lambda\right), \tag{S101}
\end{align*}
$$

and

$$
\begin{align*}
& \tilde{s}_{0}(\beta)=\tilde{h}_{0}\left(\beta e^{i \omega_{L} T}\right) \\
& \tilde{s}_{1}(\beta)=\tilde{h}_{1}\left(\beta e^{i \omega_{L} T}\right) e^{-i T \omega_{L}} \\
& \tilde{s}_{2}(\beta)=\tilde{h}_{2}\left(\beta e^{i \omega_{L} T}\right) e^{-2 i T \omega_{L}} \tag{S102}
\end{align*}
$$

we have

$$
\delta K(\mathbf{x}, \overline{\mathbf{x}})=\left|e_{b}(\beta, \bar{\beta})\right|^{2}\left(\begin{array}{l}
\tilde{s}_{0}(\mathbf{x}, \beta)+\bar{\beta} \tilde{s}_{1}(\mathbf{x}, \beta)+\bar{\beta}^{2} \tilde{s}_{2}(\mathbf{x}, \beta)  \tag{S103}\\
+ \text { c.c. } \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}) \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}, \text { c.c. })
\end{array}\right)
$$

So,

$$
\frac{\delta K(\mathbf{x}, \overline{\mathbf{x}})}{K_{0}(\mathbf{x}, \overline{\mathbf{x}})}=\left(\begin{array}{l}
\tilde{s}_{0}(\mathbf{x}, \beta)+\bar{\beta} \tilde{s}_{1}(\mathbf{x}, \beta)+\bar{\beta}^{2} \tilde{s}_{2}(\mathbf{x}, \beta)  \tag{S104}\\
+ \text { c.c. } \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}) \\
+(\mathbf{x} \leftrightarrow \overline{\mathbf{x}}, \text { c.c. })
\end{array}\right)
$$

## III. EXPERIMENTAL PROPOSALS

## A. Threshold of Kerr non-linearity

As stated in the main text, we compare the Kerr coefficient with the drive amplitude to determined if the system is in the weak or strong non-linear regime. The Kerr non-linearity effectively induces a photon number-dependent resonant frequency shift $\omega_{\mathrm{m}}^{\prime} \rightarrow\left(\omega_{\mathrm{m}}-2 K_{\text {err }} n\right)$ with $n$ being the photon number; namely, every time a photon is added into the resonator, the resonant frequency will shift by $-2 K_{\text {err }}$. If we consider an initially on-resonance drive with $\omega_{\mathrm{L}}=\omega_{\mathrm{m}}$, as more and more photons are populated in the resonator, the drive will equivalently become more and more detuned. Strong non-linearity refers to the situation that after the first photon enters the resonator, the drive detuning $\Delta_{\mathrm{L}} \approx 2 K_{\text {err }}$ becomes large enough so that additional photons at $\omega_{\mathrm{L}}$ can no longer populate the resonator. In other words, the anharmonic resonator now may be approximated as a two-level qubit with negligible excitations in its higher energy levels. The intra-cavity photon number ( $n \equiv b^{\dagger} b$ ) under a detuned drive tone is given by

$$
\begin{equation*}
n=\frac{\Omega^{2}}{\Delta_{\mathrm{L}}^{2}+\left(\frac{\kappa}{2}\right)^{2}} \tag{S105}
\end{equation*}
$$

where $\kappa=\kappa_{\mathrm{i}}+\kappa_{\mathrm{e}}$ is the full linewidth of the resonator. In our theoretical analysis, we assumed the intrinsic loss of the resonator is negligible $\kappa_{\mathrm{i}} \approx 0$ and the external coupling $\kappa_{\mathrm{e}}$ can also be kept sufficiently small. Therefore, the strong non-linearity regime corresponds to $\Omega^{2} / 4 K_{\mathrm{err}}^{2} \ll 1$, while the weak non-linearity regime corresponds to $\Omega^{2} / 4 K_{\mathrm{err}}^{2} \gg 1$.


FIG. S1. (a) Measurement scheme for the overlap between two quantum state $|\Psi\rangle$ and $|\Phi\rangle$. (b) Experimental setup of a circuit QED system for the overlap measurement between the two modes supported by non-linear (NL) resonators A and B, respectively. Three transmon qubits $\mathrm{qA}, \mathrm{qB}$, and qC are used for quantum state control and coupling of the two resonators.

## B. Experimental setup

The kernel we defined in Eq. (1) in the main text relies on the measurement of the overlap between two states $|\Psi\rangle \equiv|\psi(T)\rangle$ and $|\Phi\rangle \equiv\left|\psi\left(T^{\prime}\right)\right\rangle$. This can be realized by controlled-SWAP (CSWAP) gate using an auxiliary qubit [12]; the states of $|\Psi\rangle$ and $|\Phi\rangle$ are swapped when the qubit is in the excited $(|e\rangle)$ state, while kept unchanged when the qubit is in the ground $(|g\rangle)$ state. The CSWAP can be decomposed into two 50:50 beam-splitter (BS) operations with a controlled phase gate (CPS) in the middle (Fig. S1(a)). Therefore, if we first apply a Hadamard gate (H) to prepare the qubit to a superposition state $(|g\rangle+|e\rangle) / \sqrt{2}$, then apply the controlled SWAP gate, followed by a second Hadamard gate (H), the system state becomes

$$
\begin{align*}
|g\rangle|\Psi\rangle|\Phi\rangle & \xrightarrow{\mathrm{H}} \frac{\sqrt{2}}{2}(|g\rangle+|e\rangle)|\Psi\rangle|\Phi\rangle \xrightarrow{\mathrm{CSWAP}} \frac{\sqrt{2}}{2}(|g\rangle|\Psi\rangle|\Phi\rangle+|e\rangle|\Phi\rangle|\Psi\rangle)  \tag{S106}\\
& \xrightarrow{\mathrm{H}} \frac{1}{2}|g\rangle(|\Psi\rangle|\Phi\rangle+|\Phi\rangle|\Psi\rangle)+\frac{1}{2}|e\rangle(|\Psi\rangle|\Phi\rangle-|\Phi\rangle|\Psi\rangle) .
\end{align*}
$$

Therefore, if we measure the probability of the qubit final state in $|g\rangle$ or $|e\rangle$, we can get the overlap $|\langle\Psi \mid \Phi\rangle|^{2}$

$$
\begin{equation*}
P_{g}=\frac{1}{2}\left(1+|\langle\Psi \mid \Phi\rangle|^{2}\right), P_{e}=\frac{1}{2}\left(1-|\langle\Psi \mid \Phi\rangle|^{2}\right) . \tag{S107}
\end{equation*}
$$

Experimentally, these quantum operations and measurements can be done in circuit QED systems [13, 14]. As illustrated in Fig. S1(b), we can use two nonlinear superconducting resonators (A and B) to prepare the states $|\Psi\rangle$ and $|\Phi\rangle$. The Kerr nonlinearity can be realized by either Josephson junction elements or superconducting kinetic inductance devices [15, 16]. The drive tone $\Omega$ can be directly sent to the cavities via a microwave port. In addition, each cavity is dispersively coupled to a transmon qubit (qA and qB, respectively) for universal control and state readout. Importantly, a third transmon $q \mathrm{C}$ is simultaneously coupled to A and B to mediate a bilinear interaction $H_{\mathrm{BS}} \propto\left(g a b^{\dagger}+g^{*} a^{\dagger} b\right)$ under two microwave drive tones, which enables the beam-splitter interaction. The CPS gate can be realized by harnessing the dispersive coupling, for example, between $q B$ and cavity B. Then we will be able to infer the state overlap by measuring the population of qB.
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